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Optimum Range and Endurance of a Piston Propeller Aircraft with Cambered Wing

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An exact solution of the maximum range, and a highly accurate approximate solution of the maximum endurance, are proposed for cruising flight of the piston-powered aircraft installed with a constant-speed propeller and cambered wing. It is proven that the constant-altitude/constant-speed cruise range can be independently optimized, even for the aircraft with cambered wing, without substitution of the optimum speed of other flight regimes.

Nomenclature

C_D	=	drag coefficient
C_{D_0}	=	parasite drag coefficient
C_{D_1}, C_{D_2}	=	induced drag coefficients
C_L	=	lift coefficient
c_P	=	specific fuel consumption
D	=	drag force
$E_{\max}$	=	maximum lift-to-drag ratio
L	=	lift force
P_e	=	engine power
P_r	=	required power
P_0	=	sea level engine power
R	=	range
S	=	wing area
t	=	endurance
u	=	ratio of the speed to the initial condition minimum drag speed
V	=	airspeed
V_{md}	=	minimum drag airspeed
V_{me}	=	maximum endurance airspeed
V_{mp}	=	minimum power airspeed
W	=	weight of aircraft
W_F	=	weight of fuel
w	=	ratio of the aircraft weight to the initial condition of weight
y	=	facilitation variable
ζ	=	fuel weight fraction
η_P	=	propeller efficiency
ρ	=	air density
σ	=	relative air density
χ	=	wing camber parameter

Subscripts

BR	=	Breguet range
CE	=	constant altitude-constant speed endurance
CR	=	constant altitude-constant speed range
cr	=	critical altitude
SR	=	specific range
0	=	initial condition
1	=	final condition

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Introduction

RANGE and endurance are basic operational requirements and design criterion for almost all aircraft. For this reason, range and endurance performance of aircraft have been studied by several authors, starting with Coffin.[1] Recently, Torenbeek[2] has reviewed cruise performance and range prediction in detail, but concentrated more on turbojet aircraft performance.

The latest developments in aviation regarding usage of piston propeller aircraft, such as unmanned air vehicle (UAV) designs and small aircraft transportation system (SATS) efforts, have brought back the importance of piston–propeller aircraft performance. Preliminary or conceptual design of both UAVs and small aircraft require quick and more accurate prediction of the range and endurance. Recent studies for prediction of the range and endurance of piston–propeller aircraft were published almost 20 years ago by Hale,[3] as well as by Ojha[4] and Bert.[5]

Hale[3] introduced a constant-altitude/constant-speed cruise flight solution, but for a drag polar of a wing without camber and by using the optimum speed of constant-lift-coefficient/constant-speed flight regime as the optimum speed. Ojha[4] developed an exact solution for range and an approximate solution for endurance, but for a drag polar of wing without camber. Bert[5] attempted to calculate the optimum range for aircraft having wings with camber. However, Bert only optimized the specific range, not the range itself. All of the authors have used constant specific fuel consumption and propeller efficiency assumptions, which are almost all true for cruising flight of the piston-powered aircraft installed with constant-speed propellers.

In this paper, an exact solution of the maximum range, and a highly accurate approximate solution of the maximum endurance, are proposed for cruising flight of the piston-powered aircraft installed with constant-speed propeller and having a cambered wing because both range and endurance are important for UAVs and small general aviation aircraft, depending on the flight mission.

Fundamental Assumptions

Actual drag polars of aircraft from experimental results usually differ from the symmetric parabolic form because most aircraft have cambered wing sections. For this reason, the shifted parabolic drag polar for cambered wings

$$C_D = C_{D_0} - C_{D_1} C_L + C_{D_2} C_L^2 \tag{1}$$

will result in more accurate range and endurance performance prediction.

According to most authors, including Ananthasayanam,[6] Hale,[3] McCormick,[7] and Torenbeek,[8] the power of an aircraft piston engine is proportional to the relative density when there is no supercharging. For supercharged engines, the power of the engine is constant up to the critical altitude and proportional to the relative density after the critical altitude. Thus, for normal aspirated engines,

$$P_e \propto \sigma P_0 \tag{2a}$$

and for supercharged engines,

$$P_e \propto (\sigma/\sigma_{cr}) P_0 \tag{2b}$$

The variation of the specific fuel consumption c_P of propeller-driven aircraft shows that there is some variation, but it is advantageous to assume it to be constant not only for all speeds, but at all altitudes as well.[6] Thus,

$$c_P = \text{const} \tag{3}$$

The propeller efficiency varies with the airspeed for fixed-pitch propellers. However, with a variable-pitch constant-speed propeller η_P can be assumed to be constant over the design operating speed range.[3] Because most of the current piston–propeller aircraft are installed with constant-speed propellers, for the range and endurance analysis, it is assumed that

$$\eta_P = \text{const} \tag{4}$$

Minimum Drag and Minimum Required Power Speeds for Level Flight

The range and endurance performance of aircraft depends on the minimum drag and minimum power speeds. For the aircraft with symmetrical section wing drag polar, these characteristics are known very well by most aeronautical societies. However, they are slightly different, but also simple in the case of the aircraft with a cambered wing drag polar. The drag of the aircraft

$$D = C_D(\rho/2)V^2 S = \left(C_{D_0} - C_{D_1} C_L + C_{D_2} C_L^2\right)(\rho/2)V^2 S$$

Because

$$C_L = 2W/\rho V^2 S \tag{5}$$

for level flight, then

$$D = C_{D_0}(\rho/2)V^2 S - C_{D_1} W + C_{D_2}(2W^2/\rho V^2 S) \tag{6}$$

For minimum drag a $\mathrm{d}D/\mathrm{d}V = 0$ condition must be satisfied. Thus,

$$\frac{\mathrm{d}D}{\mathrm{d}V} = C_{D_0}\rho V S - C_{D_2}\frac{4W^2}{\rho V^3 S} = 0$$

results in minimum drag speed

$$V_{\mathrm{md}} = \left(C_{D_2}/C_{D_0}\right)^{\frac{1}{4}}(2W/\rho S)^{\frac{1}{2}} \tag{7}$$

This is equivalent to the minimum drag speed for a symmetrical section wing drag polar. However, the maximum lift-to-drag ratio where minimum drag speed occurs is different

$$E_{\max} = (L/D)_{\max} = \left[1/\left(2\sqrt{C_{D_0}C_{D_2}} - C_{D_1}\right)\right] = \left[E_{m_0}/(1-\chi)\right] \tag{8}$$

where

$$\chi = C_{D_1}/\left(2\sqrt{C_{D_0}C_{D_2}}\right) \tag{9}$$

is the wing camber parameter and $E_{m_0} = \frac{1}{2}\sqrt{(C_{D_0}C_{D_2})}$ is the maximum lift-to-drag ratio of the wing without camber.

The required power

$$P_r = DV = \left(C_{D_0} - C_{D_1} C_L + C_{D_2} C_L^2\right)(\rho/2)V^3 S$$

becomes

$$P_r = C_{D_0}(\rho/2)V^3 S - C_{D_1} W V + C_{D_2}(2W^2/\rho V S) \tag{10}$$

for level flight conditions. For minimum required power, the $\mathrm{d}P_r/\mathrm{d}V = 0$ condition must be satisfied. Therefore,

$$\frac{\mathrm{d}P_r}{\mathrm{d}V} = 3C_{D_0}\frac{\rho}{2}V^2 S - C_{D_1} W - C_{D_2}\frac{2W^2}{\rho V^2 S} = 0$$

results in minimum required power speed

$$V_{\mathrm{mp}} = \left(\chi + \sqrt{\chi^2+3}/3\right)^{\frac{1}{2}}\left(C_{D_2}/C_{D_0}\right)^{\frac{1}{4}}(2W/\rho S)^{\frac{1}{2}} \tag{11}$$

The ratio of the minimum required power speed to the minimum drag speed

$$V_{\mathrm{mp}}/V_{\mathrm{md}} = \sqrt{\left(\chi + \sqrt{\chi^2+3}\right)/3} \tag{12}$$

shows that the aircraft with cambered wing requires higher minimum power than the aircraft with symmetrical wing sections because $\chi > 0$.

Differential Equations of the Range and Endurance

Both range and endurance equations are derived from the power specific fuel consumption definition

$$c_P = \frac{\mathrm{d}W_F}{P_e\,\mathrm{d}t} \tag{13}$$

and steady-state level flight equations

$$\eta_P P_e - DV = 0, \qquad L - W = 0 \tag{14}$$

Because the change in aircraft weight is equal to the fuel weight consumed, and

$$V = \frac{\mathrm{d}R}{\mathrm{d}t} \tag{15}$$

then the substitution of Eqs. (14) and (15) into Eq. (13) results in

$$\mathrm{d}R = -\frac{\eta_P}{c_P}\frac{C_L}{C_D}\frac{\mathrm{d}W}{W} \tag{16}$$

for the range, and

$$\mathrm{d}t = -\frac{\eta_P(\rho S)^{\frac{1}{2}}}{\sqrt{2}c_P}\frac{C_L^{\frac{3}{2}}}{C_D}\frac{\mathrm{d}W}{W^{\frac{3}{2}}} \tag{17}$$

for the endurance. Depending on the flight mission requirements, UAVs and small general aviation aircraft may fly constant-lift-coefficient/constant-speed or constant-altitude/constant-speed cruise flight regimes. For this reason, optimization of both flight cases will be analyzed.

Constant-Lift-Coefficient/Constant-Speed Cruise Range and Endurance

Integration of the differential equation of range, Eq. (16), between the aircraft weight at the beginning, W_0, and the end of cruise flight, W_1, for a constant-lift-coefficient and constant-speed cruise condition gives the famous Breguet range equation

$$R_{\mathrm{BR}} = (\eta_P/c_P)(C_L/C_D)\,\ln(W_0/W_1) \tag{18}$$

Because η_P and c_P are constants in Eq. (18), then the maximum range occurs at maximum lift-to-drag ratio, which is given by Eq. (8). Therefore, the maximum cruise range for constant-lift-coefficient/constant-speed flight will be

$$R_{\mathrm{BR_{max}}} = (\eta_P/c_P)\left[1/\left(2\sqrt{C_{D_0}C_{D_2}} - C_{D_1}\right)\right]\ln(W_0/W_1) \tag{19}$$

On the other hand, integration of the differential equation of endurance, Eq. (17), for the same weight and flight condition results in

$$t_{\mathrm{BR}} = \sqrt{2}\left[\eta_P(\rho S)^{\frac{1}{2}}/c_P\right]\left(C_L^{\frac{3}{2}}/C_D\right)\left(1/\sqrt{W_1} - 1/\sqrt{W_0}\right) \tag{20}$$

For maximum endurance, $C_L^{3/2}/C_D$ must be maximum; thus, aircraft must fly with the minimum required power speed given by Eq. (11).

Then the maximum endurance for constant-lift-coefficient/constant-speed flight will be

$$t_{\mathrm{BR}_{\max}} = \frac{\sqrt{2}}{4} \frac{\eta_P (\rho S)^{\frac{1}{2}}}{c_P \left(C_{D_0} C_{D_2}^3\right)^{\frac{1}{4}}} \frac{\left(\sqrt{\chi^2+3}-\chi\right)^{\frac{3}{2}}}{1-\chi\left(\sqrt{\chi^2+3}-\chi\right)} \left(\frac{1}{\sqrt{W_1}} - \frac{1}{\sqrt{W_0}}\right) \tag{21}$$

Constant-Altitude/Constant-Speed Cruise Range and Endurance

Optimization of constant-altitude/constant-speed flight for piston-propeller aircraft was first presented by Ojha,[4] but for aircraft with symmetrical wing sections. Then Bert[5] proposed solutions for the aircraft having cambered wings. However, Bert's solution optimized only the specific range.

For simplicity, by the assumption that

$$w = W/W_0, \qquad u = V\big/V_{\mathrm{md}_0}$$

$$V_{\mathrm{md}_0} = \left(C_{D_2}\big/C_{D_0}\right)^{\frac{1}{4}} (2W_0/\rho S)^{\frac{1}{2}} \tag{22}$$

the differential equation of the range, Eq. (16), becomes

$$\mathrm{d}R = -\frac{\eta_P}{c_P} \frac{1}{\sqrt{C_{D_0} C_{D_2}}} \frac{u^2\,\mathrm{d}w}{u^4 - 2\chi u^2 w + w^2} \tag{23}$$

Integration of Eq. (23) between the aircraft weight at the beginning, W_0, and the aircraft weight at the end, W_1, of the cruise flight yields constant-altitude/constant-speed range:

$$R_{\mathrm{CR}} = \frac{\eta_P/c_P}{\sqrt{C_{D_0} C_{D_2}}} \frac{1}{\sqrt{1-\chi^2}} \left[\arctan\frac{1-\chi u^2}{u^2\sqrt{1-\chi^2}} - \arctan\frac{(1-\zeta)-\chi u^2}{u^2\sqrt{1-\chi^2}}\right] \tag{24}$$

or

$$R_{\mathrm{CR}} = \frac{\eta_P/c_P}{\sqrt{C_{D_0} C_{D_2}}} \frac{1}{\sqrt{1-\chi^2}} \arctan\frac{\zeta u^2\sqrt{1-\chi^2}}{u^4 - (2-\zeta)\chi u^2 + (1-\zeta)} \tag{25}$$

where

$$\zeta = (W_0 - W_1)/W_0 \tag{26}$$

is the cruise fuel weight fraction. For constant altitude, maximizing the range implies

$$\frac{\mathrm{d}R}{\mathrm{d}u} = 0 \tag{27}$$

Ojha[4] mentioned that the resulting equation will not yield an analytic expression for u because of the arctangent function and set the arctangent term equal to its argument by assuming that it is generally much less than unity. However, there is a much better description for Ojha's assumption.

Assume that

$$y = \frac{\zeta u^2\sqrt{1-\chi^2}}{u^4 - (2-\zeta)\chi u^2 + (1-\zeta)}$$

then

$$R_{\mathrm{CR}} = \frac{\eta_P/c_P}{\sqrt{C_{D_0} C_{D_2}}} \frac{1}{\sqrt{1-\chi^2}} \arctan y$$

$$\frac{R_{\mathrm{CR}}}{\mathrm{d}u} = \frac{\eta_P/c_P}{\sqrt{C_{D_0} C_{D_2}}} \frac{1}{\sqrt{1-\chi^2}} \frac{1}{1+y^2} \frac{\mathrm{d}y}{\mathrm{d}u} = 0$$

Because $y > 0$, then

$$\frac{\mathrm{d}y}{\mathrm{d}u} = 0 \tag{28}$$

satisfies the maximum range condition. Equation (28) results in

$$u_{\mathrm{opt}} = V_{\mathrm{opt}}\big/V_{\mathrm{md}_0} = (1-\zeta)^{\frac{1}{4}} \tag{29}$$

The optimum speed for the constant-altitude/constant-speed maximum range is equal to the optimum speed found by Ojha[4] for a symmetrical section wing drag polar because minimum drag speeds are equal both for cambered wing drag polar and symmetrical section wing drag polar. However, the optimum speed found by Eq. (29) must never be smaller than the minimum required power speed due to safety considerations described by Hale.[3] Thus,

$$\sqrt{1-\zeta} \geq \left[\left(\chi + \sqrt{\chi^2+3}\right)\big/3\right] \tag{30}$$

Substitution of Eq. (29) into Eq. (25) gives the maximum cruise range for constant-altitude/constant-speed flight:

$$R_{\mathrm{CR}_{\max}} = \frac{\eta_P/c_P}{\sqrt{C_{D_0} C_{D_2}}} \frac{1}{\sqrt{1-\chi^2}} \arctan\frac{\zeta\sqrt{1-\chi^2}}{2\sqrt{1-\zeta} - \chi(2-\zeta)} \tag{31}$$

If η_P and c_P are not dependent on altitude, then Eq. (31) shows that the maximum range does not vary with flight altitude, and it only depends on aerodynamic-propulsive characteristics and the fuel weight fraction of the aircraft. However, the optimum speed varies with altitude because of the proportionality to minimum drag speed. Thus, flight at higher altitudes will result in shorter cruise flight time for the same maximum range.

Because $\mathrm{d}R/\mathrm{d}W$ defines the specific range of aircraft, maximum specific range can be obtained at

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{u^2}{u^4 - 2\chi u^2 w + w^2}\right) = 0 \tag{32}$$

which results in

$$u_{\mathrm{SR}} = \sqrt{w} \qquad \text{or} \qquad V_{\mathrm{SR}} = V_{\mathrm{md}} \tag{33}$$

Substitution of Eq. (33) into Eq. (25) gives

$$R_{\mathrm{SR}} = \frac{\eta_P/c_P}{\sqrt{C_{D_0} C_{D_2}}} \frac{1}{\sqrt{1-\chi^2}} \arctan\frac{\zeta w\sqrt{1-\chi^2}}{w^2 - (2-\zeta)\chi w + (1-\zeta)} \tag{34}$$

If optimum speed for maximum specific range is set to conditions at the beginning of cruise flight, then $w = 1$, and

$$R_{\mathrm{SR}} = \frac{\eta_P/c_P}{\sqrt{C_{D_0} C_{D_2}}} \frac{1}{\sqrt{1-\chi^2}} \arctan\frac{\zeta}{(2-\zeta)}\sqrt{\frac{1+\chi}{1-\chi}} \tag{35}$$

The range for maximum specific range is not equal to the maximum range. Moreover, the airspeed to obtain maximum range is slower than the airspeed for maximum specific range. However, the Fig. 1 comparison of the constant-altitude/constant-speed maximum range with the range for maximum specific range shows that both conditions have closer values for lower fuel weight fractions and smaller camber parameters.

Substitution of the variables defined by Eq. (22) into the differential equation of endurance, Eq. (17), results in

$$\mathrm{d}t = -\frac{\eta_P (\rho S)^{\frac{1}{2}}}{\sqrt{2W_0} c_P} \frac{u}{\left(C_{D_0} C_{D_2}^3\right)^{\frac{1}{4}}} \frac{\mathrm{d}w}{u^4 - 2\chi w u^2 + w^2} \tag{36}$$

Integration of Eq. (36) between aircraft weights at the beginning and end of cruise flight yields constant-altitude/constant-speed endurance:

$$t_{\mathrm{CE}} = \frac{\eta_P/c_P}{\sqrt{C_{D_0} C_{D_2}}} \frac{1}{V_{\mathrm{md}_0}\sqrt{1-\chi^2}} \frac{1}{u} \arctan\frac{\zeta u^2\sqrt{1-\chi^2}}{u^4 - (2-\zeta)\chi u^2 + (1-\zeta)} \tag{37}$$

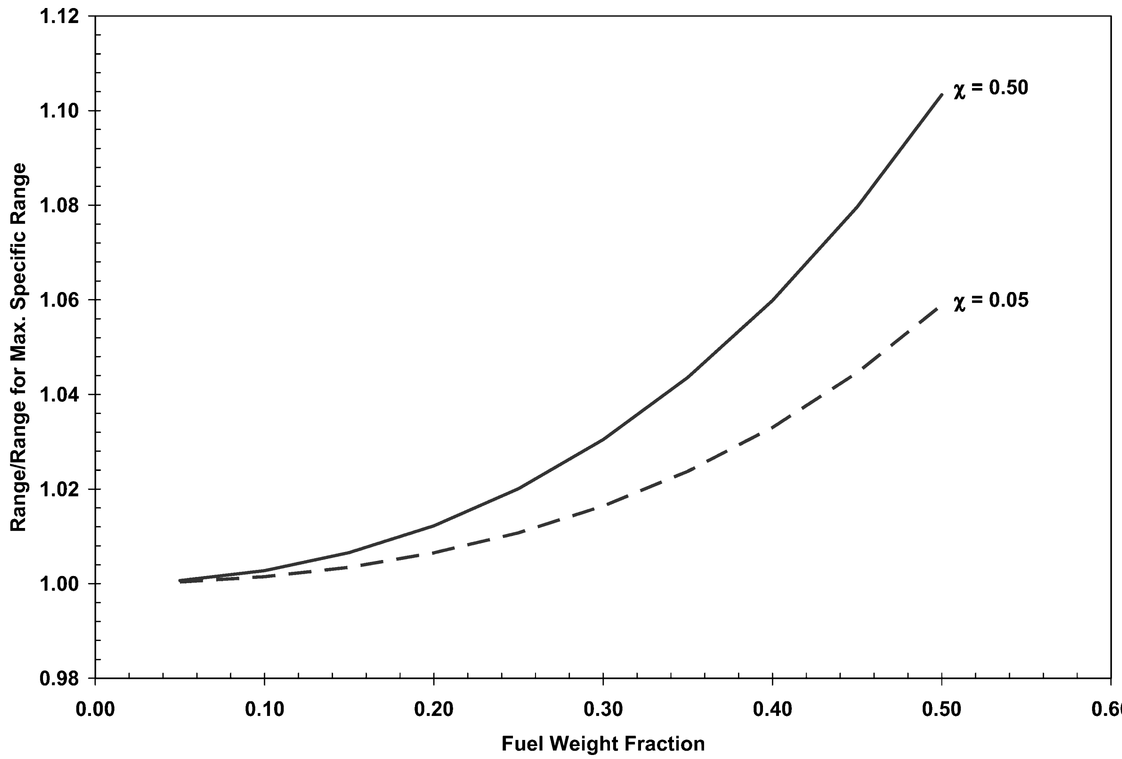

Fig. 1 Comparison of the constant-altitude/constant-speed maximum range with the range for maximum specific range.

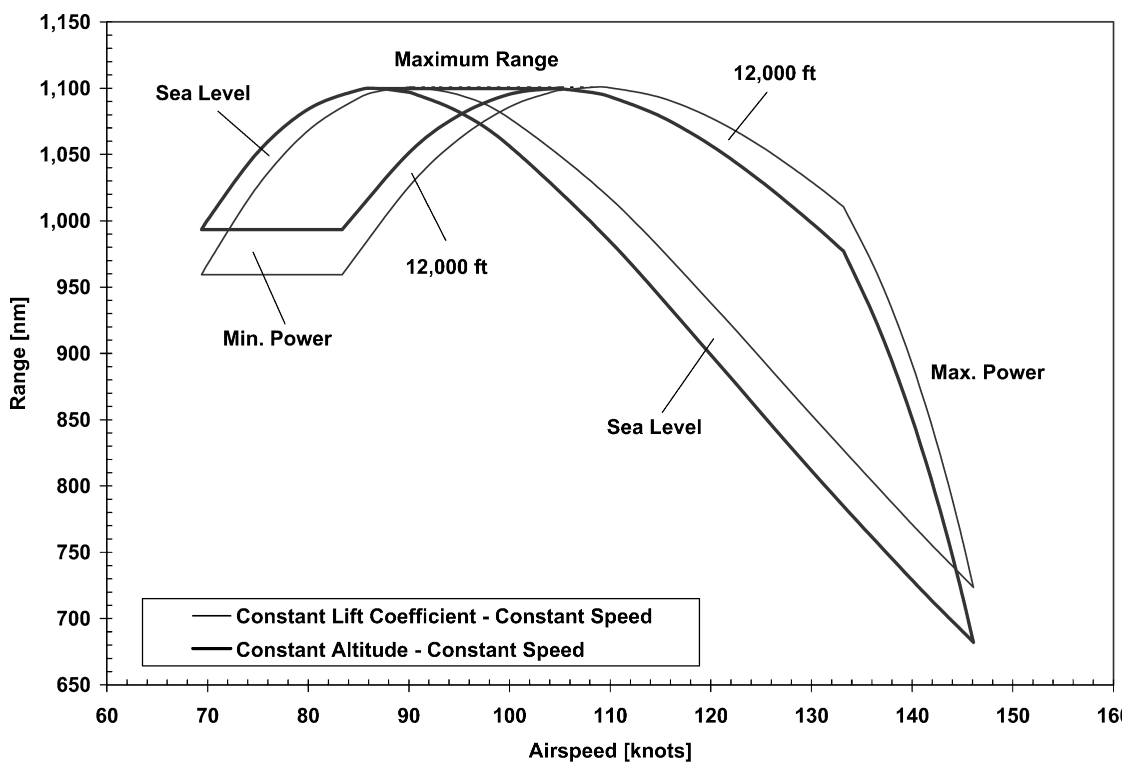

Fig. 2 Range vs speed.

For constant altitude, maximization of the endurance implies

$$\frac{\mathrm{d}t_{\mathrm{CE}}}{\mathrm{d}u} = 0 \tag{38}$$

The result obtained from Eq. (38) will not yield an analytic expression for u, because of the arctangent function and $(1/u)$ multiplier. However, now the "arctangent" term can be set equal to its argument as it was proposed by both Torenbeek[2] and Ojha.[4] This approximation results in

$$t_{\mathrm{CE}} \cong \frac{\eta_P/c_P}{\sqrt{C_{D_0}C_{D_2}}}\frac{1}{V_{\mathrm{md}_0}}\frac{\zeta u}{u^4-(2-\zeta)\chi u^2+(1-\zeta)} \tag{39}$$

Application of the maximum endurance condition given by Eq. (38) to this accurate approximation[2] results in

$$V_{\mathrm{me}}^2 \cong \frac{2-\zeta}{6}\left[\chi+\sqrt{\chi^2+12\frac{1-\zeta}{(2-\zeta)^2}}\right]V_{\mathrm{md}_0}^2 \tag{40}$$

Application

A hypothetical aircraft of weight 2650 lb (1202 kg), with a 174-ft^2 (16.17 m^2) wing area is considered. The drag polar is assumed

$$C_D = 0.0243 - 0.0036C_L + 0.0800C_L^2 \tag{41}$$

Propeller efficiency is 0.85 and power specific fuel consumption is 0.45. The cruise fuel weight fraction is assumed to be 0.14. The range and endurance of the aircraft is calculated for International Standard Atmosphere conditions.

The ranges of the aircraft for sea-level and 12,000-ft pressure altitude are plotted in Fig. 2 both for constant-lift-coefficient/constant-speed and constant-altitude/constant-speed cruise flight conditions.

Figure 2 shows that there is almost no difference between maximum ranges when two flight conditions are compared, although Eqs. (19) and (30) are analytically different. However, depending

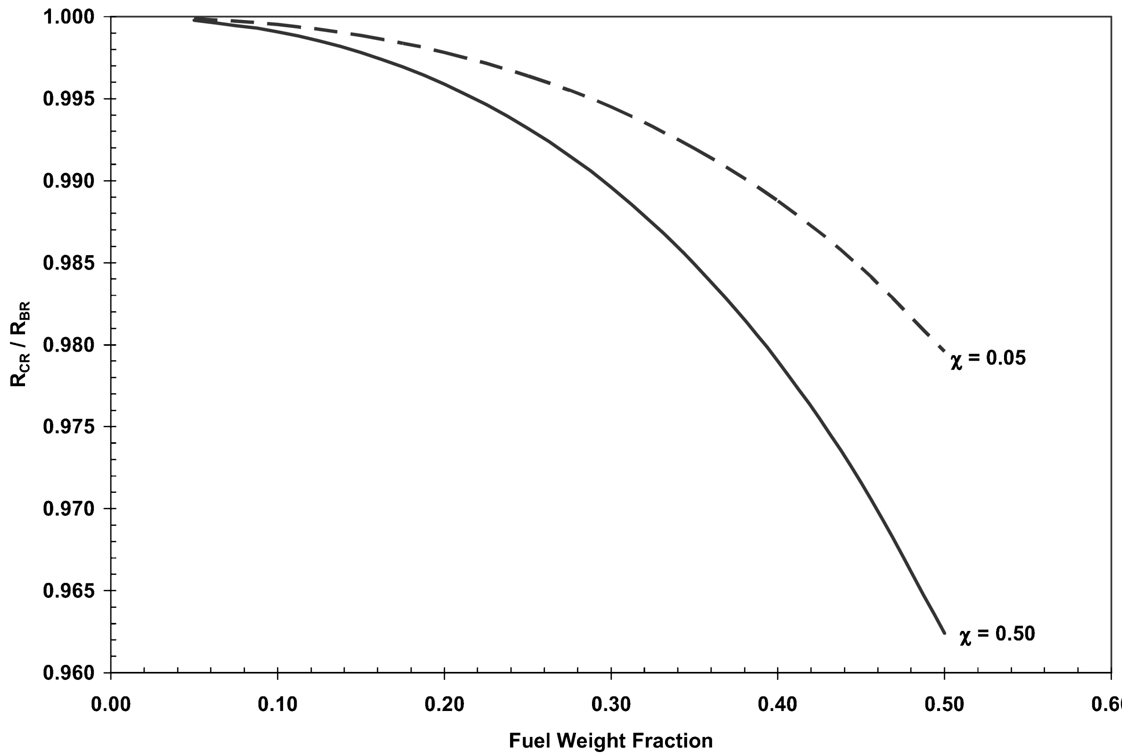

Fig. 3 Comparison of the constant-altitude/constant-speed and constant-lift-coefficient/constant-speed cruise maximum ranges for two camber parameters.

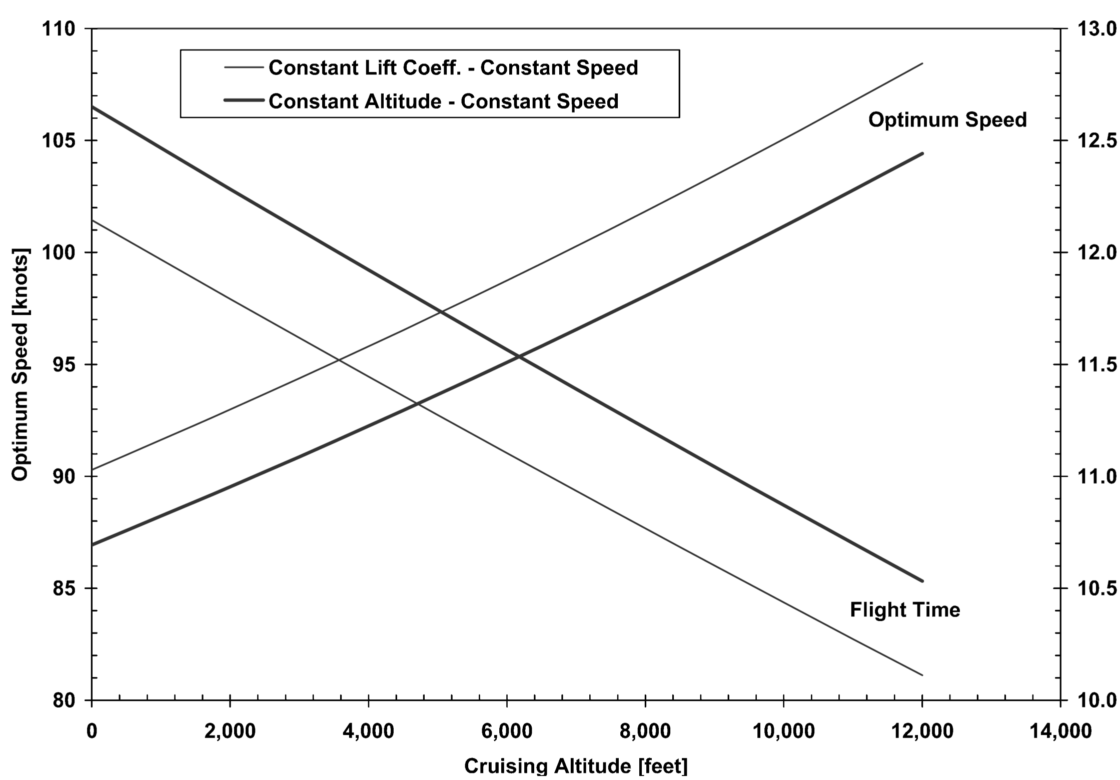

Fig. 4 Optimum speed and flight time vs altitude.

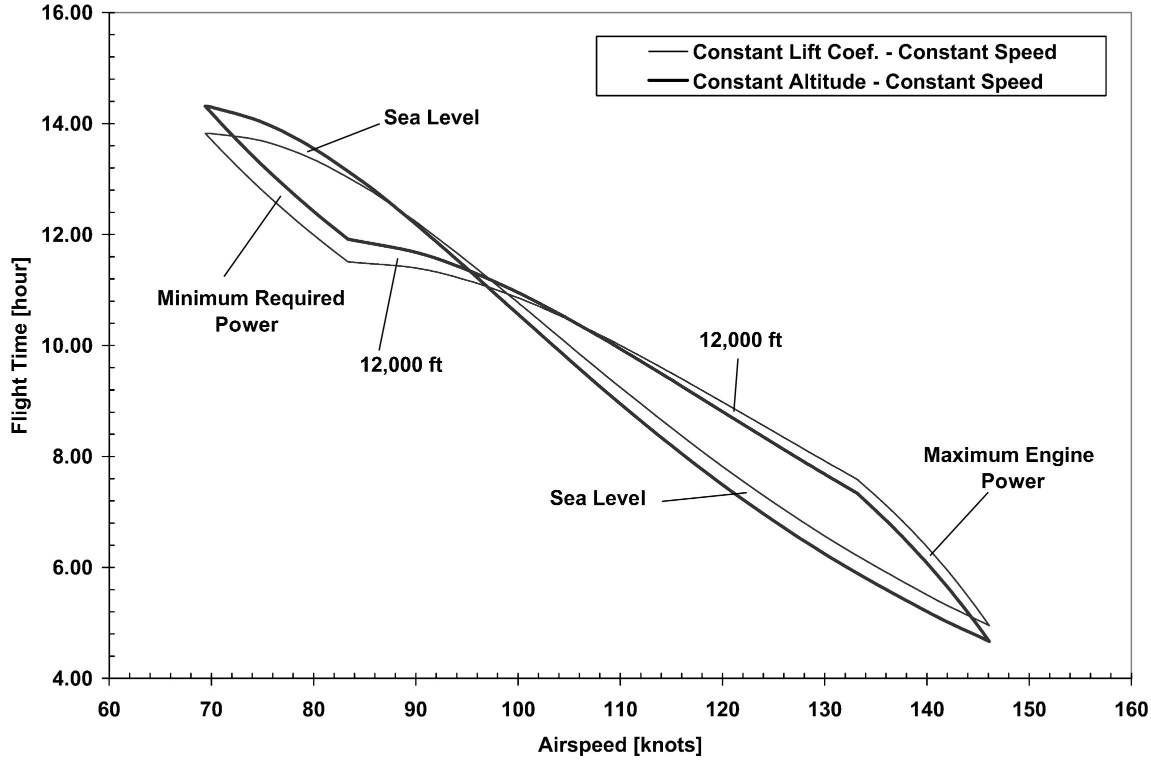

Fig. 5 Flight time vs speed.

on the cruise fuel weight fraction and wing camber parameter, differences between maximum ranges of two flight conditions increase, as seen in Fig. 3, although they never become more than 5%.

As the theory denotes, the constant-altitude/constant-speed cruise flight regime has slower optimum speeds than the constant-lift-coefficient/constant-speed regime, and this leads to longer flight times for reaching maximum range. This is shown in Fig. 4 for the hypothetical aircraft.

Variation of the flight time with speed is plotted in Fig. 5 for cruise flights at sea level and 12,000 ft. Figure 5 shows that constant-altitude/constant-speed cruise flight regime provides longer maximum endurance at minimum required power speed flight when compared to the constant-lift-coefficient/constant-speed cruise flight regime.

Conclusions

The constant-altitude/constant-speed cruise range can be independently optimized, even for the aircraft with cambered wing and without substitution of the optimum speed of other flight regimes. Although maximization of the specific range results in closer maximum range values, it is proven that maximization of the specific range does not yield maximum range. Both maximum cruise range and maximum endurance are dependent on the cruise fuel weight fraction and the camber parameter. Constant-lift-coefficient/constant-speed and constant-altitude/constant-speed cruise flight regimes result in slightly different cruise ranges and endurance for smaller fuel weight fractions and camber parameters. However, the differences become significant for greater values of the cruise fuel weight fraction and camber parameter.

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